Is an Equilateral Triangle Isosceles? Student Perspectives

Penelope Currie and John Pegg
Centre for Cognitive Research in Learning and Teaching
University of New England

Identified relationships between figures, based on their properties, is an important characteristic of Level 3 thinking in the van Hiele theory. However, there has been no research directed at how such relationships, leading to class inclusion, evolve. This study, involving indepth interviews with 24 secondary students, addresses this issue by considering their attempts at grouping seven different triangle types. The results reveal important features about perceived relationships between figures, and the existence of a developmental path.

The van Hiele theory (van Hiele, 1986), which consists of five levels of cognitive development provides a framework from which to view students' thinking processes in Geometry. The third level of thinking is characterised by the identification of relationships between properties of known figures which acknowledge the dependence of properties upon each other, and the existence of a network of relations between these figures (Pegg, 1995). The description of Level 3 highlights the notion of class inclusion as an essential characteristic at this stage of cognitive growth. Van Hiele (1986, p.95) described this aspect as requiring a student to "build up a network of relations in which the figures are interconnected on the basis of their properties".

Studies have highlighted that ideas of class inclusion are not easily grasped, and their acquisition remain an important hurdle to be overcome before formal deduction (Mayberry, 1981; Fuys, Geddes and Tischler, 1985; and, Pegg and Davey, 1991). In the light of this, it is surprising that no study has specifically targeted the notion of class inclusion for investigation. This paper addresses this by exploring, via task-orientated interview sessions, students' attempts at identifying relationships between different triangles.

Two research questions guided the study. They are:
1. What was the nature of the links students formed when grouping different triangles?
2. Was there evidence of some developmental pattern in the different responses?

Design

This paper reports the results of one aspect of a larger study developed to explore aspects of class inclusion. Twenty-four students, six from each of Year 8 to Year 11, were selected from two secondary schools in Armidale. The students were of above average ability and there were equal numbers of males and females.

The purpose of this study was to have the students identify and justify relationships among seven different triangle types, namely, acute scalene, obtuse scalene, right scalene, acute isosceles, obtuse isosceles, right isosceles, and equilateral. The format of the interview is contained in Figure 1 and includes student tasks and the questioning focus common to each interview. This format was chosen as it enabled the students to work with familiar recalled information, supplemented information, individual tree designs, and discussion involving prompts and probes from the interviewer. The continual revisiting of the same relationships, as drawn on different maps by the students, provided a vehicle for extracting further information, as the maps were used as a catalyst for discussion concerning the reasons for the existence of links (or relationships). The investigation required the analysis of responses to questions (ii), (iv), and (v) which deal with the relationships between the seven triangle types. The analysis of the responses required the development of a diagrammatical summary which combines the information gathered in student maps and interview transcript form.

Triangle Relationships

(i) Int: I would like you to write a list of all the triangle names you can think of. Begin with acute angled scalene. Draw each triangle.
(ii) Int: Design a tree diagram which links the different triangles. Draw a sketch to link each type.
(discussion follows concerning the reasons for links and/or lack of links)
(the following three points are addressed if required)
(iii) Int: There are some triangles that we can add to this list. (provide triangles not recalled)
Draw a sketch of each new triangle.
(iv) Int: Design a second tree diagram incorporating all the triangles on the list.
(discussion follows concerning the reasons for links and/or lack of links)
(v) Int: Return now to your first map. I would like you to add the new triangles to your original tree.
(discussion follows concerning the reasons for links and/or lack of links)

Figure 1: Summary Interview Structure

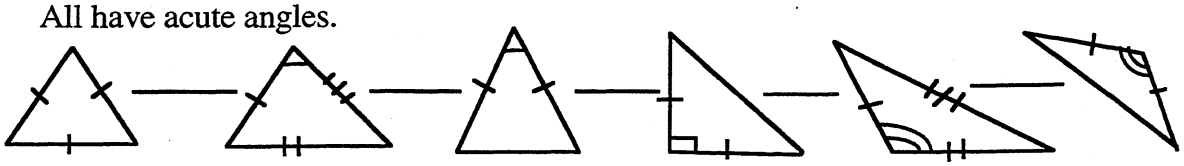
Results

Overall, the students found the ideas familiar, but the questions were seen as non-routine. There were several types of responses identified. The codings take into consideration the types of relationships described and the justification of these relationships. Below is a detailed analysis of the main categories of responses including a description of one typical student response for each type incorporating a diagrammatical representation of their response and relevant samples taken from the interview transcript.

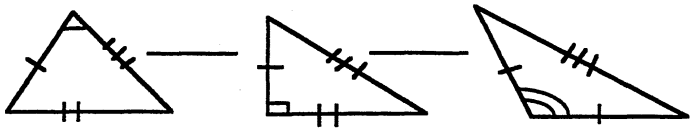
Type A

This type of response indicates the use of a single similar feature or property to relate the triangles together, such as containing acute angles, unequal sides, or equal sides. Only one feature or property is used in each grouping, and the groupings change according to the property or feature that is the focus. Hence, a class of shapes has not formed an identity of its own. There was only one response coded as Type A and this is summarised in Figure 2 below.

All have acute angles.



All have unequal sides.



All have at least two sides equal.

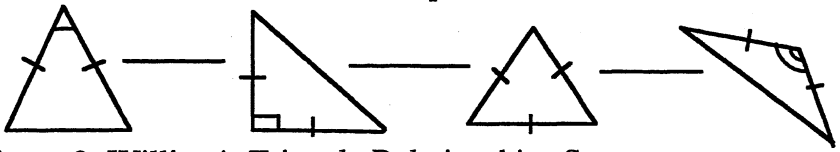


Figure 2: William's Triangle Relationships Summary

The following excerpt conveys the justifications for the relationships based on similar features or properties described by William. It illustrates that the relationships form spontaneously, and groupings change as often as the identifying property or feature changes. There were no links formed across the groups unless the student was prompted.

Int: What is that link there for? (right angled isosceles to obtuse angled scalene)
William: Because it has got acute angles.
Int: Anything else?
William: Um no. [pause] There I have put all the unequal sides together.

Int: No more links there?
William: No...
Int: What have you done up on this top row?
William: They all have three sides and they all have at least one angle that is an acute angle.
Int: What have you done on the next row?
William: Um they all have uneven sides.
Int: And here?
William: They have, all have, at least two sides the same.

Overall, a Type A response forms groups of triangles based on a single similar feature or property. In William's case the features were acute angles, unequal sides, and having at least two sides equal. The groupings change as frequently as the identifying characteristic, without the formation of dominant relationships. Triangles can belong to a number of groups. Hence, classes of triangles are not a workable identity.

Type B

These responses include the formation of three mutually exclusive classes of triangles, these being, scalene, isosceles, and equilateral. Each of these classes represents a unit which has a specific name to encapsulate the similarities of the group. The similarities include one, or a combination of the following; side properties, angle properties, and axes of symmetry. None of the eight responses within this type makes a link between the class of isosceles and the equilateral triangle.

Megan's response (see Figure 3) represents the best of a Type B response as it includes the three classes (standard for this Type) and relationships that exist due to angle-type links. Megan's justifications for classes of shapes are based on the following similar properties: (i) the equilateral triangle has three sides and three angles equal; (ii) the isosceles class of triangles has two sides equal, two angles equal, and one line of symmetry; and (iii) the scalene triangles have no sides equal and no angles equal.

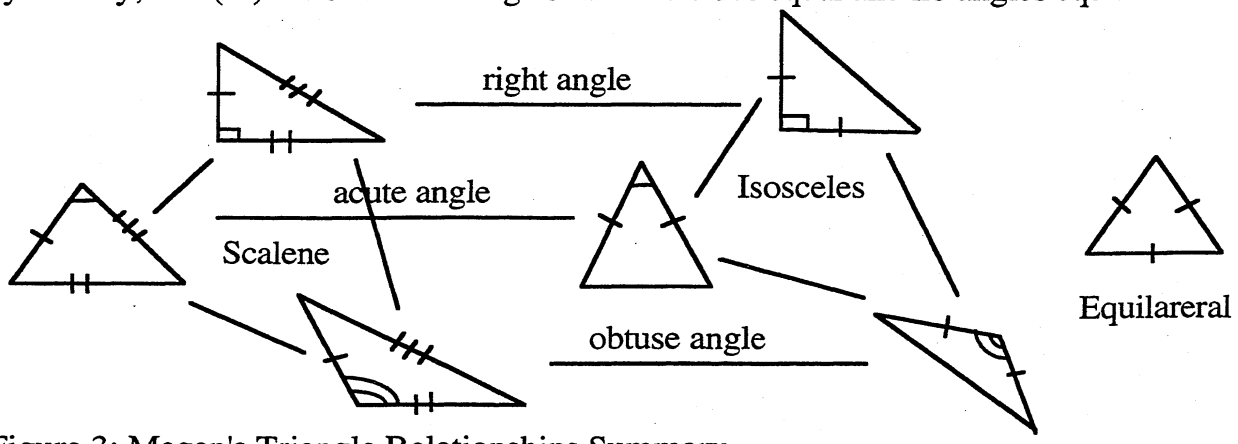


Figure 3: Megan's Triangle Relationships Summary

A link is not made from the isosceles class to the equilateral class based on acute angles as the equilateral is described as having specifically three angles equal and therefore it is not possible to link the classes for any reason. The links across classes are only made according to angle types with the exception of the equilateral. This is illustrated by the following excerpt.

Megan: ... The equilateral triangle is on its own ...
Int: Do you think that the isosceles and the equilateral belong together?
Megan: Oh no not really.
Int: Why don't they?
Megan: Because that one (one side of isosceles triangle) there can be different. The only thing that they have in common is that they have some equal sides and equal angles. An isosceles only has two equal sides and two equal angles whereas an equilateral has three equal sides and three equal angles. With these scalene triangles each side can be all different lengths and I think that sometimes they can be right angled and the same as the isosceles (can also be right angled).

Megan: Um this one is a right angled scalene um I don't want that link between the equilateral and the isosceles. Those isosceles are linked because they both have a line of symmetry.

In summary, Type B responses all contain three triangle-type classes, i.e., scalene, isosceles, and equilateral. The name of each class carries meaning and a triangle must contain all the characteristics of the class, and no others, to belong to that class. Three categories differentiate between those responses with completely isolated classes and those that describe angle-type links between classes. Category 1 comprises of triangle-type classes only. Category 2 comprises of three triangle-type classes and the addition of right-angle links. Category 3 comprises of three triangle-type classes and the addition of right-angle, acute-angle, and obtuse-angle links. None of the responses in Type B make a link to the equilateral triangle.

Type C

These responses are consistent with Type B responses described above and hence, a summary diagram is not provided. The small difference is based upon the addition of tentative links made between the equilateral triangle and the isosceles class of triangles. The similarities are noted between these triangle types but the differences identified do not allow the link to be made. Three responses were coded as Type C.

The discussion concerning the appropriateness of constructing a link between the equilateral and the isosceles classes indicates the awareness that the equilateral triangle also has two sides equal and/or two angles equal, however, a link is not made based on the equilateral triangle containing three sides and/or three angles equal. Mark's discussion illustrates the formation of three triangle-type classes and indecisive comments concerning a possible link between the equilateral and the isosceles triangle.

Mark: So I can link up the scalenes because they have no sides that are equal. Um, the equilateral um, it can be linked to there, it is not really linked it is sort of tacked on the end.
Int: Do you think that link there exists? (isosceles to equilateral)
Mark: Well I can't see how it can link I should cross that one out.
Int: So why are these others linked? (isosceles triangles)
Mark: They all have two sides that are equal.

When questioned again about the appropriateness of a link between the equilateral and the isosceles class of triangles, Mark described both the similarity and the differences between the classes, and ended with an inability to make a decision concerning the relationship.

Int: Now you still have the equilateral linked on there. What do you think about that?
Mark: Um, since these are both 45 degrees there and no they are all 60 degrees on that one, um these turn out to be equal, um I don't know maybe two angles are equal on both of them.

Overall, the responses in this group contain the three triangle-type classes and angle-type relationships across classes with the exception of the equilateral triangle. The distinguishing feature of these responses is that the similarities between the equilateral triangle and the isosceles triangles are noted but the differences identified do not allow a link to be made.

Type D

This group of responses made definite links between the equilateral and isosceles triangle classes based on similar properties. Eight responses were coded as Type D. Each of the responses in this group include three triangle-type classes with the addition of a link between the isosceles and equilateral classes. The relationship is based on both classes containing two equal sides and/or two equal angles, although, the equilateral triangle is not yet described as a subset of the isosceles class of triangles.

Beth's response as summarised in Figure 4 is typical of a Type D response including three angle-type links. She described the triangle-type classes in terms of similar side properties. The link between the equilateral and the isosceles class of triangles exists on the basis of both classes containing two sides the same length. Beth also acknowledged that the equilateral triangle is acute, and links the equilateral triangle with the acute isosceles triangle for this reason.

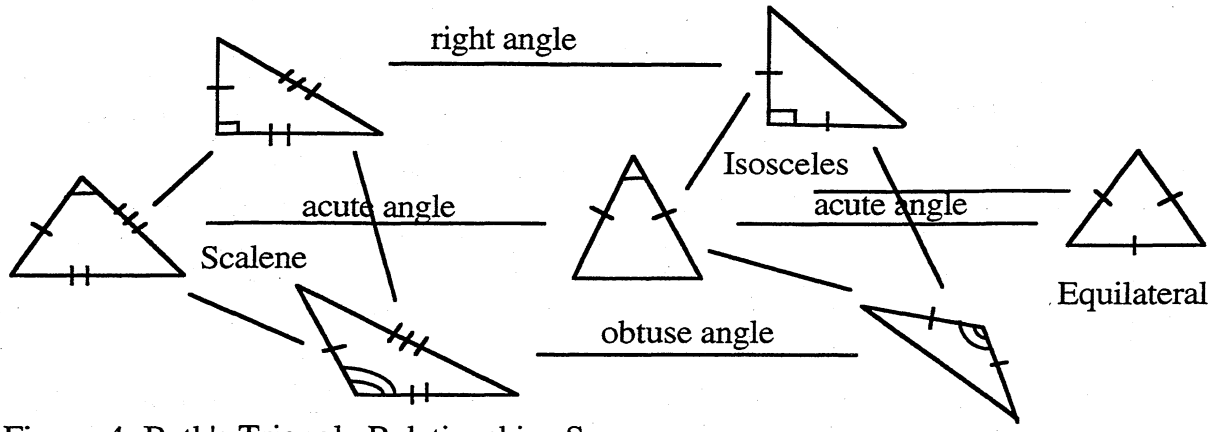


Figure 4: Beth's Triangle Relationships Summary

Int: And why do they (isosceles triangles) link to your equilateral?
Beth: Because they (angles on equilateral triangle) are all acute.
Int: Any other reason why the equilateral links to the isosceles?
Beth: Um because it has got two of the sides the same on both of them. I can link that with that because they are all acute angled.

In summary, Type D responses make a significant link between the equilateral and the isosceles class of triangles based on similar properties. At no stage are equilateral triangles described as a subset of the isosceles class of triangles but a relationship exists between the two classes within Type D. The four categories differ from each other through the gradual addition of angle-type links across the scalene, isosceles and equilateral classes. Category 1 contains no angle-type links. Category 2 acknowledges the existence of angle-type links in discussion but does not allow them to cross the established triangle-type classes. Category 3 contains right-angle links across triangle-type classes only. Finally, Category 4 adds relationships based on right-angle, acute-angle, and obtuse-angle similarities across the three triangle-type classes.

Type E

This group of responses is consistent with a Type D response, however, they include statements concerning the equilateral triangle's relationship to the isosceles class as not based on similar properties only, but due to the equilateral triangle being a form of an isosceles triangle. There are two responses coded within this category. Although both responses include this information there are differences in the responses which have placed them into two categories. Category 1 suggests that the equilateral triangle may be a subset of the isosceles triangle but it is not yet fully accepted. Category 2 accepts this notion but does so without justification.

Nathan's response is characteristic of a Type D Category 4 coding with one differing element. The distinguishing feature that separates this response from Type D responses is Nathan's suggestion that the equilateral triangle may be a subset of the isosceles triangle class, although this has not yet been accepted. This is represented diagrammatically by the dotted boundary in Figure 5.

When Nathan was asked to provide a reason for the link between the acute isosceles and the equilateral triangle he began by explaining that both triangles have acute angles. Nathan paused before continuing and went on to explain that both classes of triangles do have two equal sides. When prompted to place the equilateral within the isosceles class of triangles Nathan restated the similarity between them but was hesitant to accept the equilateral triangle as a subset of the isosceles class of triangles.

Int: Now why do you have those two (equilateral and isosceles) linked?
Nathan: Because they both have acute angles. Um [pause] That is because they have all got acute angles.
Int: Any other reason why the equilateral and the isosceles link?
Nathan: Because on those two the sides are equal like that. Yeah they are both isosceles I think.

Int: So do you think the equilateral is isosceles?
Nathan: Yes I think so. It has got two equal sides like an isosceles.

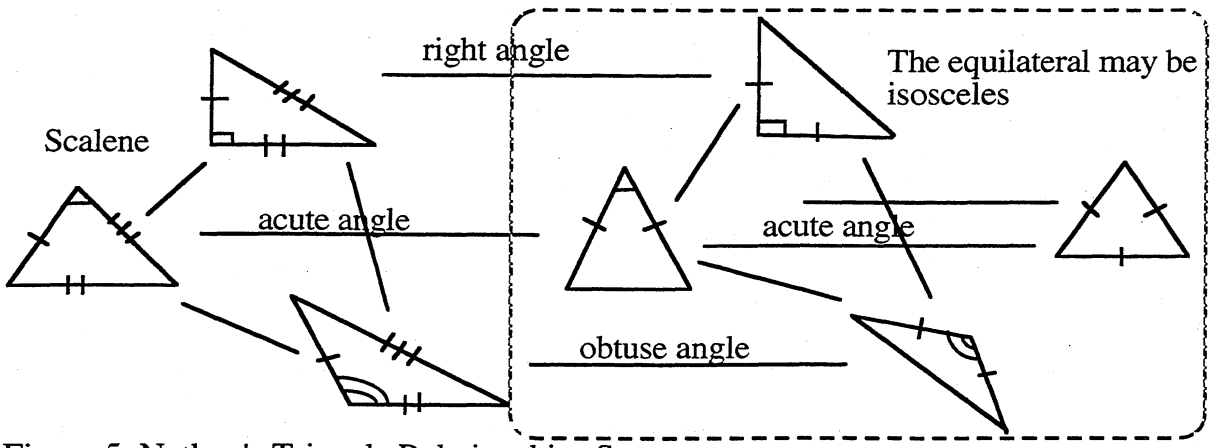


Figure 5: Nathan's Triangle Relationships Summary

Overall, both responses contain relationships between the triangles and justifications similar to the Type D responses. The difference observed is the ability to discuss the possibility that the equilateral is a subset of the isosceles class of triangles. Although Nathan was hesitant to accept the notion, his attempt at justifying this relationship sets the response apart from a Type D categorisation.

Type F

A typical Type F response describes the isosceles class of triangles as containing a subset, namely, the equilateral triangle. The equilateral triangle is identified as a form of the isosceles triangle, and the student is able to justify the equilateral triangle's existence within this class. It can also be argued why an equilateral is an isosceles triangle but that the isosceles triangle is not an equilateral. There was one response coded as Type F. The diagram developed from the information given by Sally is contained in Figure 6.

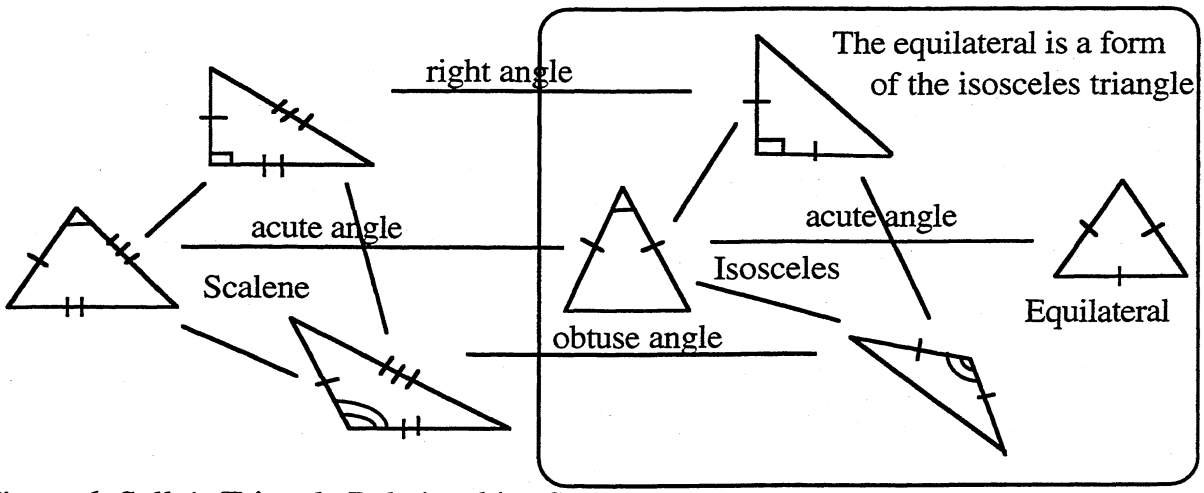


Figure 6: Sally's Triangle Relationships Summary

Sally's explanation of the tree diagram illustrates the clearly defined relationships that exist between the classes of triangles. Sally included the notion of class inclusion as an important feature of her tree design and was able to justify this on the basis of properties.

Sally: Um, all the triangles begin the tree. Then I differentiate between the side length with two or more equal sides and sides are not equal.
Int: So you have ended up with the equilateral and the isosceles on the same branch. Do you see those two triangles linking?
Sally: In that they have equal sides and equal angles. You could say that the equilateral triangle is a form of the isosceles triangle in that it does have two equal angles and two equal sides.

Sally acknowledged the angle-type relationships in her discussion and explained that it is another method of grouping the triangles. Sally went on to explain that an important element of the design is to make sure that the equilateral class branches from the isosceles branch. This concept has become a dominating feature of the relationships between the different triangles.

Sally: I have done much the same thing but I think that it works better in that it includes the equilateral and the isosceles on the same branch.

In summary, the Type F response includes subsets within classes based on similar properties. The equilateral is clearly identified as a form of isosceles triangle. The student is able to justify this class inclusion and argue that the converse is not true.

Type G

This type of response also makes explicit that the equilateral is a subset of the isosceles class of triangles. The difference lies in the acquisition of further conditions upon these subsets. There is one response coded as Type G and the relationships are summarised in Figure 7.

Adam explained that the equilateral is a "special isosceles" based on properties, namely, two equal sides. Adam developed the relationships further as the link between the acute isosceles and the equilateral becomes significant. A greater link is acknowledged between these two triangles based on the equilateral triangle's existence within the isosceles class and containing acute angles.

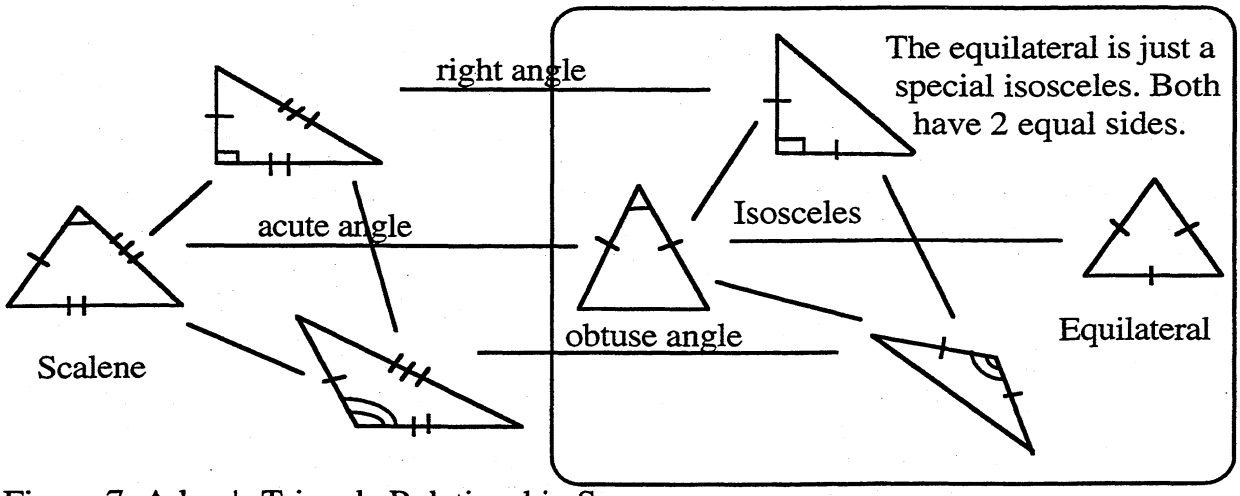


Figure 7: Adam's Triangle Relationship Summary

Adam: The equilateral can be linked to the isosceles.
Int: How come?
Adam: Because these triangles both have two equal sides and the equilateral is just a special isosceles. Some right angled triangles can be isosceles. [pause] ...
Int: What is the main reason these are linked together? (isosceles and equilateral)
Adam: They have the same things that these have.
Int: Do you see a link to all those?
Adam: No just the acute ones. Because 60 is less than 90.

Figure 7 developed as a summary of Adam's response illustrates the difference between the Type G and Type F response. Although Sally provided more detailed justifications for the relationships described, she did not acknowledge the significance of the relationship between the acute isosceles triangle and the equilateral triangle. Adam stated that the equilateral triangle is a subset of the isosceles class of triangles and made a major link to the acute isosceles when considering concurrently angle-type links, triangle-type classes, and the notion of class inclusion.

Summary

The students' responses were coded into seven types. These can be summarised as:

Type A A single similar feature is identified to relate triangles. Triangles appear in more than one group depending upon the identifying feature for each group.

Type B Scalene, isosceles and equilateral classes of triangles are formed and they are characterised by name and related by similar properties. No link is made between the isosceles and equilateral classes. The three categories separate responses on the addition of angle-type links across the isosceles and scalene classes.

Type C Three triangle-type classes are formed with angle-type links across classes. Similar properties are noted between the isosceles and equilateral triangles but differences described do not allow the link to be made.

Type D Relationships are formed across the equilateral and isosceles classes of triangles based on similar properties, but the equilateral triangle is not regarded as a subset of the isosceles class. The four categories within this type separate the responses on the addition of angle-type links across the three classes of triangles.

Type E Similar to a Type D response with indecisive statements concerning the possible inclusion of the equilateral triangle within the isosceles class of triangles, or statements concerning this notion of class inclusion without justification.

Type F The equilateral triangle is a subset of the isosceles class of triangles with justification based on properties.

Type G Similar to Type F but the subsets formed acquire further conditions. The relationship between the acute angled isosceles and the equilateral triangle becomes significant and can be justified.

Conclusion

The results presented in this paper provide a detailed description of student understandings of class inclusion as it relates to different triangle groups. Van Hiele (1986) described class inclusion as Level 3 thinking but implies that this notion is developed once all Level 2 properties become a totality. The student responses provide evidence which reveals that class inclusion is not a 'hit or miss' notion. Instead, there appears to be a developmental path which enables students to gradually acquire this skill. Further research is needed to explore this aspect with respect to quadrilaterals, and then to see if some general framework may help interpret the categories identified.

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